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Jackson-Type Theorems for Approximation with Hermite–Birkhoff Interpolatory Side Conditions

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1. INTRODUCTION

In this paper we obtain estimates for the cost of Hermite–Birkhoff interpolatory side conditions, placed on uniform approximation by trigonometric polynomials.

Given a positive integer κ; a finite set $t_1, \ldots, t_\gamma$ of distinct points in $-\pi \leqslant t < \pi$; for each $i = 1, \ldots, \gamma$, a nonempty subset K_i of the set $\{0, 1, \ldots, \kappa\}$; and $f \in C^{*\kappa}[-\pi, \pi]$, define the set of κ times continuously differentiable 2π periodic functions; the set $A_\kappa = \{ g \in C^{*\kappa}[-\pi, \pi]: g^{(j)}(t_i) = f^{(j)}(t_i); j \in K_i ; i = 1, \ldots, \gamma\}$. Let N_ν be the space of trigonometric polynomials of degree not exceeding ν. For each $\nu = 0, 1, 2, 3, \ldots$, define

$$E_\nu(f) = \inf_{g \in N_\nu} \| f - g \|, \tag{1.1}$$

where

$$\| f - g \| = \sup_{-\pi \leqslant t \leqslant \pi} | f(t) - g(t) |. \tag{1.2}$$

Similarly define $e_\nu(f)$ as the infimum of (1.2) over those g in N_ν with constant part zero; and if $N_\nu \cap A_\kappa$ is nonempty, $E_\nu(f, A_\kappa)$ as the infimum of (1.2) over g in $N_\nu \cap A_\kappa$.

We show that $E_\nu(f, A_\kappa)$ satisfies an estimate of the Jackson type appropriate for κ times continuously differentiable functions. That is, $E_\nu(f, A_\kappa) = O(e_\nu(f^{(\kappa)})/\nu^\kappa)$. Comparing this estimate of $E_\nu(f, A_\kappa)$ with $E_\nu(f)$ we show that for all $f \in C^{*\kappa}[-\pi, \pi]$

$$e_\nu(f^{(\kappa)})/\nu^\kappa = O(E_\nu(f)^{1-\epsilon}), \qquad \text{for all} \quad \epsilon > 0.$$

On the other hand, given any sequence of positive numbers $\{h_\nu\}_{\nu=1}^{\infty}$, increasing without bound, and an A_κ including at least one derivative constraint, we can construct an $f \in C^{*\kappa}[-\pi, \pi]$ such that

$$\limsup_{\nu \to \infty} E_\nu(f, A_\kappa)/h_\nu E_\nu(f) \geqslant 1.$$

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2. The Main Theorem

THEOREM 2.1. *For each* $\kappa = 1, 2, 3,\ldots$ *there exists an* $M_\kappa > 0$, *and for each set of side conditions* A_κ, *a* $\nu_1 = \nu_1(\kappa, t_1, \ldots, t_\gamma)$, *not depending on* f, *such that for any* $f \in C^{*\kappa}[-\pi, \pi]$, $E_\nu(f, A_\kappa)$ *exists and satisfies*

$$E_\nu(f, A_\kappa) \leqslant M_\kappa e_\nu(f^{(\kappa)})/\nu^\kappa$$

for all ν *greater than* ν_1.

Proof. We need the following version of one of the standard Jackson theorems. (For the standard theorem, see for example, Cheney [2, pp. 145–146].)

LEMMA 2.2. *For all positive integers* κ, *there exists a positive constant* C_κ, *and for any* $f \in C^{*\kappa}[-\pi, \pi]$, *a sequence of trigonometric polynomials* $\{T_\nu: T_\nu \in N_\nu\}$ *such that*

$$\|(f - T_\nu)^{(j)}\| \leqslant C_\kappa(1/\nu^{\kappa-j})\, e_\nu(f^{(\kappa)}); \qquad j = 0, 1, \ldots, \kappa; \qquad \nu = 1, 2, 3, \ldots.$$

Proof. Let j_ν be the Jackson kernel normalized so that

$$\int_{-\pi}^{\pi} j_\nu(t)\, dt = 1. \tag{2.1}$$

Write

$$J_\nu(f, x) = \int_{-\pi}^{\pi} f(x + t)\, j_\nu(t)\, dt$$

It is well known that there exists an $M > 0$ such that

$$\|f - J_\nu(f)\| \leqslant (M/\nu)\, \|f^{(1)}\|, \tag{2.2}$$

for all $f \in C^{*1}[-\pi, \pi]$. The proof now proceeds by induction.

Induction basis. Let t_ν be the best approximation to $f^{(\kappa)}$ from N_ν, with constant part zero. Let $P(g)$, $g \in C^*[-\pi, \pi]$, be the indefinite integral of g such that $\int_{-\pi}^{\pi} P(g) = 0$. Let P^κ be the κ-wise composition of operators P, and $S_\nu = P^\kappa(t_\nu)$. Then

$$\|f^{(\kappa)} - S_\nu^{(\kappa)}\| = e_\nu(f^{(\kappa)}), \qquad \nu = 1, 2, 3, \ldots .$$

Induction step. If for some $m = 0, 1, \ldots, \kappa - 1$ and some $C_m > 0$, there exists a sequence of trigonometric polynomials $\{S_\nu: S_\nu \in N_\nu\}$ such that

$$\|(f - S_\nu)^{(\kappa-j)}\| \leqslant C_m \nu^{-j} e_\nu(f^{(\kappa)}); \qquad j = 0, \ldots, m; \qquad \nu = 1, 2, 3, \ldots;$$

then there exists $C_{m+1} \leqslant C_m(M+2)$ such that

$$\|(f - S_\nu - J_\nu(f - S_\nu))^{(\kappa-j)}\| \leqslant C_{m+1}\nu^{-j}e_\nu(f^{(\kappa)}); \qquad j = 0,..., m+1; \quad \nu = 1, 2, 3,... .$$

Proof. Use the identity

$$J_\nu^{(\kappa-j)}(f - S_\nu) = J_\nu((f - S_\nu)^{(\kappa-j)}).$$

Now the induction step for $j = m + 1$ follows from the Jackson theorem (2.2); and that for $j = 0,..., m$ is a consequence of $\| J_\nu \| = 1$.

Proof of Theorem 2.1. Let T be the unit circle. Let f; t_i, $i = 1,..., \gamma$; K_i, $i = 1,..., \gamma$ satisfy the conditions of Theorem 2.1; and let $\{T_\nu\}$ be a sequence of trigonometric polynomials providing the estimate of Lemma 2.2.

By the Hausdorff property of T we can find disjoint open sets $B_1,..., B_\gamma$ in T containing $t_1,..., t_\gamma$, respectively. Urysohn's theorem now guarantees the existence of functions $f_j \in C(T)$, $j = 1,..., \gamma$, such that

$$\begin{aligned} f_j(t_j) &= 1, & \\ 0 \leqslant f_j(t) &\leqslant 1, & t \in B_j, \\ f_j(t) &= 0, & t \in T\backslash B_j. \end{aligned}$$

By the SAIN property of trigonometric approximation in conjunction with point evaluations, Deutsch and Morris [3; Theorem 4.1], there exists a ν_2 such that for $\nu \geqslant \nu_2$ there exist approximations $q_{\nu j}$ from N_ν to the f_j satisfying

$$\| q_{\nu j} \| = 1,$$
$$q_{\nu j}(t_i) = f_j(t_i), \qquad i = 1,..., \gamma; \qquad j = 1,..., \gamma,$$

and if $\delta_\nu = \max_{j=1,...,\gamma} \| q_{\nu j} - f_j \|$, then

$$\lim_{\nu\to\infty} \delta_\nu = 0. \tag{2.3}$$

Let $\lambda = [\nu/(\kappa + 1)]$, $\lambda_1 = [\lambda/(\kappa + 1)]$, where $[\cdot]$ is the integral part function, and let ν_3 be so large that $\lambda_1 \geqslant \max(\nu_2, 1)$. Suppose throughout the following that $\nu \geqslant \nu_3$. Note

$$\lambda^j \leqslant \nu^j \leqslant (2(\kappa + 1)\lambda)^j, \qquad j = 1,..., \kappa. \tag{2.4}$$

Take

$$h_{ij} = (q_{\lambda_1,i})^{\kappa+1}(\sin \lambda(t - t_i))^j, \qquad j = 0,..., \kappa; \ i = 1,..., \gamma.$$

Then

$$\| h_{ij} \| \leqslant 1, \tag{2.5}$$

$$h_{ij}^{(r)}(t_e) = 0; \qquad r = 0,..., \kappa; \ e \neq i, \tag{2.6}$$

$$h_{ij}^{(r)}(t_i) = 0, \qquad r < j, \tag{2.7}$$

and

$$h_{i,j}^{(j)}(t_i) = j!\,\lambda^j. \tag{2.8}$$

Also by the Bernstein inequality, (2.5), and (2.4)

$$\| h_{ij}^{(k)} \| \leqslant \nu^k \leqslant (2(\kappa + 1)\lambda)^k, \qquad k = 1, 2,\dots . \tag{2.9}$$

Now fix i. Let $j_1 ,\dots, j_p$ be the members of K_i in ascending order. We seek a linear combination of $h_{i0} ,\dots, h_{i\kappa}$ which will correct the values of $T_\nu^{(j)}(t_i)$, $j \in K_i$ to the $f^{(j)}(t_i)$. From (2.7), we seek a solution $\mathbf{b}$ to the equation

$$\begin{bmatrix} h_{ij_1}^{(j_1)}(t_i) & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ h_{ij_1}^{(j_p)}(t_i) & & \cdots & h_{ij_p}^{(j_p)}(t_i) \end{bmatrix} \begin{bmatrix} b_{j_1} \\ \\ b_{j_p} \end{bmatrix} = \begin{bmatrix} (f - T_\nu)^{(j_1)}(t_i) \\ \\ (f - T_\nu)^{(j_p)}(t_i) \end{bmatrix} \tag{2.10}$$

Dividing the kth row of the matrix above, and the kth element of the product vector by $j_k!\,\lambda^{jk}$; and using (2.8) the equation may be written

$$\begin{bmatrix} 1 & & & \\ a_{21} & 1 & & \\ a_{p1} & & a_{p,p-1} & 1 \end{bmatrix} \begin{bmatrix} b_{j_1} \\ \\ b_{j_p} \end{bmatrix} = \begin{bmatrix} c_{j_1} \\ \\ c_{j_p} \end{bmatrix}. \tag{2.11}$$

Since the matrix $A = (a_{ke})$ above is lower triangular and has determinant 1 a solution exists. By (2.9) there exists an M, depending only on κ such that

$$| a_{ke} | \leqslant M, \qquad k = 1,\dots, p, \qquad e = 1,\dots, p.$$

By Lemma 2.2 there exists an L depending only on κ such that

$$| c_{j_k} | \leqslant Le_\nu(f^{(\kappa)})/\nu^\kappa, \qquad k = 1,\dots, p.$$

Employing Cramer's rule,

$$| b_{j_k} | \leqslant (\kappa + 1)!\, M^\kappa Le_\nu(f^{(\kappa)})/\nu^\kappa; \qquad k = 1,\dots, p. \tag{2.12}$$

Writing $H_i = \sum_{k=1}^{p} b_{j_k} h_{i,j_k}$, and using (2.12),

$$| H_i(t)| \begin{array}{ll} \leqslant D_\kappa e_\nu(f^{(\kappa)})/\nu^\kappa, & t \in B_i , \\ \leqslant D_\kappa \delta_{\lambda_1} e_\nu(f^{(\kappa)})/\nu^\kappa, & t \in T\backslash B_i , \end{array} \tag{2.13}$$

where

$$D_\kappa = (\kappa + 1)!(\kappa + 1)\, M^\kappa L.$$

The analysis above holds for $i = 1,\dots, \gamma$. Also since by (2.6)

$$H_i^{(r)}(x_e) = 0, \qquad e \neq i, \qquad r = 0,\dots, \kappa,$$

we can find $H_1, \dots, H_\gamma$ separately, by the above, and

$$H = T_\nu + \sum_{i=1}^{\gamma} H_i$$

will belong to A_κ, the set of functions satisfying the interpolatory side conditions. It remains to estimate $\|f - H\|$; using (2.13) we find

$$|(f - H)|(t) \leqslant |f - T_\nu|(t) + \left|\sum_{i=1}^{\gamma} H_i\right|(t)$$

$$\leqslant \begin{cases} C_\kappa \dfrac{e_\nu(f^{(\kappa)})}{\nu^\kappa} + D_\kappa \dfrac{e_\nu(f^{(\kappa)})}{\nu^\kappa}(1 + (\gamma - 1)\,\delta_{\lambda_1}); & t \in \bigcup_{i=1}^{\gamma} B_i, \\ C_\kappa \dfrac{e_\nu(f^{(\kappa)})}{\nu^\kappa} + \gamma\delta_{\lambda_1} D_\kappa \dfrac{e_\nu(f^{(\kappa)})}{\nu^\kappa}; & t \in T \setminus \bigcup_{i=1}^{\gamma} B_i. \end{cases}$$

Thus

$$\|f - H\| \leqslant (C_\kappa + 2D_\kappa)(e_\nu(f^{(\kappa)})/\nu^\kappa), \qquad \nu \geqslant \nu_1,$$

where $\nu_1 \geqslant \nu_3$ is chosen so that $\delta_{\lambda_1} \leqslant 1/\gamma$, $\nu \geqslant \nu_1$. This concludes the proof.

Consider now the approximation of $f \in C^\kappa[-1, 1]$ by algebraic polynomials satisfying Hermite–Birkhoff interpolatory side conditions. Redefining A_κ, $E_\nu(f)$, and $E_\nu(f, A_\kappa)$ appropriately, Platte [5, Theorem 2.3.1] has shown

THEOREM 2.3. *If $f \in C^\kappa[-1, 1]$ then there exists a constant C, independent of ν, such that*

$$E_\nu(f, A_\kappa) \leqslant CE_{\nu-\kappa}(f^{(\kappa)}).$$

The following corollary to Theorem 2.1 dramatically improves this estimate in many cases. In fact, letting $g(\theta) = f(\cos\theta)$ and using the fact that the error in approximating g by trigonometric polynomials of degree not exceeding ν equals the error in approximating f by algebraic polynomials of degree not exceeding ν, the corollary and Lemma 3.1 show

$$E_\nu(f, A_\kappa) = O(E_\nu(f)^{1-\epsilon}) \qquad \text{for all} \quad \epsilon > 0$$

whenever the corollary applies.

COROLLARY 2.4. *For each $\kappa = 1, 2, 3,\dots,$ there exists an $M_\kappa > 0$; and for* *each set of side conditions* A_κ, *provided that* $-1 < t_i < 1$; $i = 1,\ldots,\gamma$; *a* ν_1, *not depending on f, such that for any* $f \in C^\kappa[-1, 1]$, $E_\nu(f, A_\kappa)$ *exists and satisfies*

$$E_\nu(f, A_\kappa) \leqslant M_\kappa e_\nu(g^{(\kappa)})/\nu^\kappa$$

for all ν *greater than* ν_1, *where* $g \in C^{*\kappa}[-\pi, \pi]$ *is defined by* $g(\theta) = f(\cos\theta)$.

Proof. Writing $g(\theta), g^{(1)}(\theta),\ldots, g^{(\kappa)}(\theta)$ in terms of $f(x),\ldots, (d^\kappa/dx^\kappa) f(x)$

$$\begin{bmatrix} g(\theta) \\ g^{(1)}(\theta) \\ \\ \\ g^{(\kappa)}(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdot & \cdots & 0 \\ & -\sin\theta & & & \vdots \\ & & +\sin^2\theta & & 0 \\ & & & & \vdots \\ & & & & (-\sin\theta)^\kappa \end{bmatrix} \begin{bmatrix} f(x) \\ (df/dx)(x) \\ \vdots \\ \\ (d^\kappa f/dx^\kappa)(x) \end{bmatrix}$$

we see that the matrix involved is invertible, $x \neq \pm 1$, and therefore $g(\theta)$, $g^{(1)}(\theta),\ldots, g^{(\kappa)}(\theta)$ are uniquely determined by $f(x),\ldots, (d^\kappa f/dx^\kappa)(x)$, $x \neq \pm 1$, and vice versa. Thus to the algebraic interpolation conditions correspond trigonometric interpolation conditions of the same order κ. To each node t_i of the algebraic problem, there correspond two nodes θ_{2i-1}, θ_{2i} of the trigonometric where

$$0 < \theta_{2i-1} < \pi \qquad \text{and} \qquad \theta_{2i} = -\theta_{2i-1}. \tag{2.14}$$

We apply Theorem 2.1 and find a ν_1 and a sequence $\{T_\nu\}_{\nu=\nu_1}^\infty$ satisfying the trigonometric interpolation conditions, such that

$$\| g - T_\nu \| \leqslant (M_\kappa/\nu^\kappa)\, e_\nu(g^{(\kappa)}). \tag{2.15}$$

Since the interpolation conditions occur in pairs (see (2.14)) the even functions $\tilde{T}_\nu$ given by $\tilde{T}_\nu(\theta) = (T_\nu(\theta) + T_\nu(-\theta))/2$ satisfy them also. Since g is even (2.15) implies

$$\| g - \tilde{T}_\nu \| \leqslant (M_\kappa/\nu^\kappa)\, e_\nu(g^{(\kappa)}). \tag{2.16}$$

Let $p_\nu(x) = \tilde{T}_\nu(\cos^{-1} x)$. As discussed previously p_ν satisfies the interpolation conditions of the algebraic problem. Also (2.16)

$$\| f - p_\nu \| = \| g(\cos^{-1} x) - \tilde{T}_\nu(\cos^{-1} x)\| \leqslant (M_\kappa/\nu^\kappa)\, e_\nu(g^{(\kappa)}).$$

Since $\tilde{T}_\nu$ is an even trigonometric polynomial of degree not exceeding ν, $p_\nu(x)$ is an algebraic polynomial of degree not exceeding ν. This concludes the proof.

3. Comparison of $E_\nu(f)$ and $E_\nu(f, A_\kappa)$

The question of a direct comparison of $E_\nu(f)$ and $E_\nu(f, A_\kappa)$, as opposed to a comparison of $e_\nu(f^{(\kappa)})/\nu^\kappa$ and $E_\nu(f, A_\kappa)$ remains. Below we show results in two opposing directions.

LEMMA 3.1. *If* $f \in C^{*\kappa}[-\pi, \pi]$, $\kappa \geqslant 1$, *then for all* $\epsilon > 0$

$$e_\nu(f^{(\kappa)})/\nu^\kappa = O(E_\nu(f)^{1-\epsilon}).$$

Proof. Either f has only a finite number, k, of continuous derivatives or f has an infinite number of continuous derivatives.

In the first case, using the well-known Jackson and Bernstein theorems (see, e.g., Butzer and Nessel [1, Corollary 2.2.4; Theorem 2.3.4]) characterizing the rate at which $E_\nu(f)$ goes to zero in terms of the order of magnitude of the second modulus $\omega_2(f^{(k)}, \delta)$, defined by

$$\omega_2(f, \delta) = \sup_{|h| \leqslant \delta} \| f(o + h) + f(o - h) - 2f(o) \|$$

we find *either*

(i) $e_\nu(f^{(k)}) = o(1)$ but $E_\nu(f)\, \nu^{k+\epsilon}$ is unbounded for all $\epsilon > 0$;

or

(ii) there exists α, $0 < \alpha \leqslant 1$, such that $e_\nu(f^{(k)}) = O(\nu^{-\alpha})$ but $E_\nu(f)\, \nu^{k+\alpha+\epsilon}$ is unbounded for all $\epsilon > 0$. In either case

$$(1/\nu^k)\, e_\nu(f^{(k)})/E_\nu(f) = O(\nu^\epsilon) \qquad \text{for all} \quad \epsilon > 0,$$

and since $\nu = O(E_\nu(f)^{-1/k})$ this implies

$$(1/\nu^k)\, e_\nu(f^{(k)}) = O(E_\nu(f)^{1-\epsilon}), \qquad \text{for all} \quad \epsilon > 0.$$

The desired result follows as $e_\nu(f^{(\kappa)}) = O(e_\nu(f^{(k)})/\nu^{k-\kappa})$.

If f has an infinite number of continuous derivatives, and letting T_ν be the best approximation to f from N_ν, then for $p = 1, 2, \ldots$,

$$\| f^{(p)} - T_\nu^{(p)} \| = O(E_\nu(f)^{1-\epsilon}) \qquad \text{for all} \quad \epsilon > 0.$$

This follows from a modification of the argument of Platte [5, Theorem 2.3.3]. Briefly fixing ϵ, $1 > \epsilon > 0$, and p, write

$$\begin{aligned} \| f^{(p)} - T_\nu^{(p)} \| &\leqslant \sum_{n=\nu}^{\infty} \| T_{n+1}^{(p)} - T_n^{(p)} \| \\ &\leqslant 2 \sum_{n=\nu}^{\infty} (n+1)^p E_n \\ &\leqslant \left\langle 2 \sum_{n=\nu}^{\infty} (n+1)^p E_n^{\epsilon} \right\rangle E_\nu^{1-\epsilon}, \end{aligned}$$

where the term in angular brackets is bounded since

$$E_n(f) = O(1/n^k), \qquad k = 1, 2, 3, \ldots .$$

This concludes the proof.

We also have the following, showing that we cannot have any inequality of the form

$$E_\nu(f, A_\kappa) = O(G(\nu)\, E_\nu(f)),$$

where $G(\nu)$ does not depend on f. The proof is an adaptation to the trigonometric case of the argument of Lorentz and Zeller [4].

LEMMA 3.2. *Given any sequence $\{h_\nu\}_{\nu=1}^{\infty}$ of positive numbers, and a set of interpolatory side conditions A_κ ($\kappa \geqslant 1$) including at least one constraint on $f^{(\kappa)}$, there exists $f \in C^{*\kappa}[-\pi, \pi]$ such that*

$$\limsup_{\nu\to\infty} E_\nu(f, A_\kappa)/h_\nu E_\nu(f) \geqslant 1.$$

Proof. We assume, without loss of generality, that the constraint on $f^{(\kappa)}$ is at $\theta = 0$. If ($\kappa \geqslant 1$) is odd we take $g_i = \sin(i\theta)$, $i = 1, 2, 3, \ldots$; if κ ($\geqslant 1$) is even take $g_i = \cos(i\theta)$. Given any $b > 0$ we can clearly choose an N such that

$$\sum_{i=1}^{N} i^\kappa / N \geqslant b. \tag{3.1}$$

Now with $H = \sum_{i=1}^{N} g_i/N$

$$| H^{(\kappa)}(0)| \geqslant b, \qquad \| H \| \leqslant 1. \tag{3.2}$$

Take

$$b_\nu = 2\nu^\kappa(h_\nu + 1), \qquad \nu = 1, 2, \ldots, \tag{3.3}$$

and $N_0 = 1$. Given N_{j-1} ($j \geqslant 1$), there exists, according to (3.2), a polynomial, f_j, such that

$$| f_j^{(\kappa)}(0)| \geqslant b_{N_{j-1}}, \qquad \| f_j \| \leqslant 1. \tag{3.4}$$

We denote the degree of this polynomial by N_j.

The function f of the theorem will be given by the series

$$f = \sum_{j=1}^{\infty} c_j f_j ,$$

where the $c_j > 0$ satisfy

$$c_j \leqslant j^{-2} M_j^{-1}, \qquad M_j = \max(\| f_j \|, \ldots, \| f_j^{(\kappa)} \|), \tag{3.5}$$

and

$$\sum_{j=\nu+1}^{\infty} c_j \leqslant c_\nu \| f_\nu \|. \tag{3.6}$$

For instance, we can define the numbers c_i inductively by means of the relation

$$c_i = \min\{\tfrac{1}{2}c_{i-1} \| f_{i-1} \|,..., 2^{(1-i)}c_1 \| f_1 \|, i^{-2}M_i^{-1}\}, \qquad i = 2, 3,... .$$

Note that (3.5) implies f is κ times continuously differentiable. Let

$$F_\nu = \sum_{i=1}^{\nu} c_i f_i .$$

Clearly

$$E_{N_{\nu-1}}(f) \leqslant \| f - F_{\nu-1} \| = \left\| \sum_{i=\nu}^{\infty} c_i f_i \right\|, \qquad \nu = 2, 3,...,$$

and using (3.6)

$$E_{N_{\nu-1}}(f) \leqslant 2c_\nu \| f_\nu \|. \tag{3.7}$$

Let Q be any trigonometric polynomial of degree not exceeding $N_{\nu-1}$ such that $Q^{(\kappa)}(0) = f^{(\kappa)}(0)$.

Writing

$$\| Q - f \| \geqslant \| Q - F_{\nu-1} \| - \| F_{\nu-1} - f \|$$

it follows using Bernstein's inequality that

$$\| Q - f \| \geqslant c_\nu \mid f_\nu^{(\kappa)}(0) \mid / N_{\nu-1}^\kappa - 2c_\nu \| f_\nu \|,$$

and by (3.3), (3.4) that

$$\| Q - f \| \geqslant 2c_\nu h_{N_{\nu-1}} .$$

Since Q was an arbitrary polynomial subject to $Q^{(\kappa)}(0) = f^{(\kappa)}(0)$ it follows that

$$E_{N_{\nu-1}}(f, A_\kappa) \geqslant 2c_\nu h_{N_{\nu-1}} \tag{3.8}$$

(3.7) and (3.8) together imply

$$E_{N_{\nu-1}}(f, A_\kappa)/E_{N_{\nu-1}}(f) \geqslant h_{N_{\nu-1}} , \qquad \nu = 2, 3,...;$$

the desired result.

Note added in proof. Further results regarding approximation by algebraic polynomials satisfying Hermite-Birkhoff interpolatory side conditions can be found in R. K. BEATSON, "Degree of Approximation Theorems for Approximation with Side Conditions," Dissertation, University of Canterbury, Christchurch, New Zealand, 1978.

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